The pseudo mathematics of Thomas Cool

J.B. van Rongen, 2012-03-03

I was introduced into the world of Thomas Cool's mathematics through a number of internet sites. He has written various books and papers that he published privately. The topics are diverse: econometrics, statistics, mathematics and the teaching of mathematics. Neither books nor papers have received much attention from the mathematical community in the Netherlands. I will give a brief review of the following three documents, with emphasis on the errors in his reasoning:

- [1] *A Logic of Exceptions (2007):* proposes to use a three-valued logic: (True, False, Nonsense) [2007];
- [2] *Conquest of the Plane (2011):* about teaching maths [2011b] - reviewed in Euclides [2012];
- [3] *Contra Cantor, pro Occam (also 2011)*: can be seen as an expanded chapter 11 of [1] where it is "demonstrated" that the reals are countable [2011a].

All three can be found on his website (`http://www.dataweb.nl/~cool/`). I am referring to the version that I downloaded on 2012-02-28. [1] has the interesting but not so new idea that we should look at a three valued logic. Besides the traditional *true* and *false* we should look at a third category: the *"exception"* in the title. The author (=Thomas Cool) himself calls is the *"Nonsense"* category.

This is not a rigorous treatment of the subject. After a lengthy but not very formal introduction to standard logic, it looks at various well known "paradoxes" and introduces these exceptions around page 128. Here he introduces sets that reference itselves in the definition as a means to avoid paradoxes. From there on it degrades from a simple introduction into logic into a pseudo mathematical mess. See section 1.

[2] is more about teaching maths than about maths, nevertheless it is full of half truths and errors. It is instructive to show what is wrong in one particular paragraph. See section 2 and the more general review by Jeroen Spandaw [2012].

In [3] the author tries to refute Cantor's diagonal argument and "proves" that the reals are countable. Maybe [1] and [2] can be seen as educational expositions that do not need the full rigour of mathematics, but this is an argument in the foundation of mathematics and I don't think the topic is even tought at all in Dutch secundary schools. We will have a look at it in section 3. Interestingly the author has removed this article from the main page of his site since my first e-mail to him where I criticised it.

1 Using a self referencing non-set solves all "paradoxes"

Richard Gill wrote a slightly favourable review of [1] in Gill [2008]. I do not know whether he reviewed the same version as I downloaded in 2012. That can be the problem with self published material. It looks very unlikely that he overlooked the following error that occurs several times in the book. I take the first example of it that I found. The same faulty reasoning can be found in [3].

On page 128, still in his treatise of the classical two-valued logic, the author looks at this set:

$$Z := \{y | y \notin y\} \tag{1}$$

Here y is any set. Z is the Russell paradoxal set (a set that is the set of all sets that do not contain themselves). The author tries to avoid this paradox by introducing the following self-referential set definition:

$$Z^* := \{y | y \notin y \land y \in Z^*\} \tag{2}$$

This is not a proper definition. Let's make it a little bit more clear and start with a set A, and a property $P(a)$ on each element of $a \in A$ that evaluates true or false. Define a new set similar to Z^*:

$$\Psi := \{y \in A | P(y) \land y \in \Psi\} \tag{3}$$

This is not a definition of a unique set. Assume a set Ψ^* satisfies the definition. Then:

$$\Psi^* = \{y \in A | P(y) \land y \in \Psi^*\} = \{y \in A | P(y)\} \cap \{y \in A | y \in \Psi^*\} \tag{4}$$

From this we conclude that Ψ^* can be every subset of $\{y \in A | P(y)\}$ and therefore that (2) and (3) are not proper definitions.

The author has tried to "fix" his argument on 2012-03-03 by replacing (2) with $Z{*} := \{y| \left((y \neq Z{*}) \Rightarrow (y \notin y)\right) \lor \left((y \in Z{*}) \Rightarrow (y \notin y \land y \in Z{*})\right)\}$. Of course this does not make things any better.

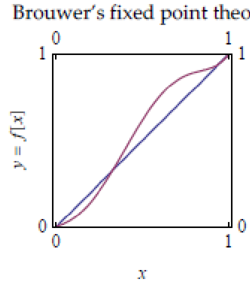
2 Using a non-theorem has a high "educational" value.

Instructive for the way the author works is the following paragraph copied from [2]. See also Spandaw [2012]:

12.1.1 The notion of a fixed point

Consider function *f* from *x* on the interval [0, 1] to *y* on the interval [0, 1]. A special function is $y = x$, also known as the diagonal. When we consider an arbitrary continuous function that starts at {0, 0} and that ends at {1, 1} then this function must touch or cross the diagonal at least once (not counting the origin). Below graph uses an arbitrary function. This example is smooth but it might have kicks and bounds, and cross more times, as long as it is continuous. A point where it touches or crosses the diagonal is called a *fixed point*: there $f[x] = x$, or the application of the function to that point generates that very point itself again. Repeated application still gives the same point. This is an instance of Brouwer's fixed point theorem.

- Brouwer's fixed point theorem on the unit interval.



We apply this notion now to functions. We collect all the functions that we know into something called "function space" $\mathcal{F}$. Instead of addition we have the addition of derivatives and instead of multiplication we have the product and chain rules. Instead of the diagonal we have the "identity function" - a function is always identical to itself. Now we reach an important conclusion: Given Brouwer's general theorem this function space $\mathcal{F}$ must also have fixed points. In particular: there must exist a function *f* that is its own derivative, $f' = f$.

The author uses excentric notations for $f(x)$ and (a, b) but let's skip that. It says that an arbitrary continuous function f with $f(0) = 0$ and $f(1) = 1$ has a point in common with the diagonal "not counting the origin". The drawing at least suggest that he does not mean the endpoint $(1, 1)$ but some fixed point in the middle. That fixed point need not exists: $f(x) = x^2$ is one of many counterexamples. I asked the author what he meant here, and he said that he included (1,1) as a fixed point, and the whole idea was that this is meant as something instructive. He gave no further explanation. Strange.

It is clear that he does not understand *Brouwer's theorem* to the full: that a continuous mapping of $[0, 1]$ onto itself has a fixed point. He has reduces it to a trivial case with the extra conditions. Then he goes on to "use" the same theorem in a function space, where *"Brouwer's general theorem"* is not true at all. Brouwer's theorem has many generalisations (Banach spaces come to mind), but it is always necessary to have a compact and convex (sub)space that is mapped. That is not the case here.

What also strikes me about this example, is that to me it looks quite unimportant when (in the classroom) and how to derive that $\frac{d}{dx}e^x = e^x$. So what value does it have to show first that a function must exist for which $f' = f$?

3 Using a non-limit to show the reals are countable

This is an error I pointed out very early in my correspondence with the author. He calls an operation on an infinite series of sets $A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots$ a *bijection by limit* or *abstraction*. I have asked for a proper definition but he does not seem to understand that question. He gives examples, not definitions. The example he gives is true and simple: $N[0] = \{0\}$, $N[1] = \{0, 1\}$, $N[2] = \{0, 1, 2\}$, etc., thus $N[0] \subset N[1] \subset N[2] \subset \cdots \to \mathcal{N} = 0, 1, ...$ However the following "generalisation" is wrong:

3.2.2 Definition of $\mathbb{R}$

The main point resides with how we define "real numbers". Let us actually define the real numbers and proceed from there. It suffices to look at the points in [0, 1] (and others could be found by 1 / x etcetera). Let d be the number of digits:

For d = 1, we have 0.0, 0.1, 0.2, ..., 0.9, 1.0.

For d = 2, we have 0.00, 0.01, 0.02, ...0.09, 0.10, 0.11, 0.12, ..., 0.98, 0.99, 1.00.

For d = 3, we have 0.000, 0.001, ..., 1.000

Etcetera. Values in $\mathbb{N}$ can be assigned to these using this algorithm: For d = 1 we assign numbers 0, ... , 10. For d = 2 we find that 0 = 0.0 = 0.00 and thus we assign 11 to 0.01, 12 to 0.02, etcetera, skipping 0.10, 0.20, 0.30, ... since those have already been assigned too. Thus the rule is that an assignment of 0 does not require a new number from $\mathbb{N}$. Thus for real numbers with a finite number of digits d in $\mathbb{R}$ we associate a finite list of numbers in $\mathbb{N}$.

Subsequently, we let $d \to \infty$. This creates both $\mathbb{R}$ and a map between that $\mathbb{R}$ and $\mathbb{N}$.

Many mathematicians seem to regard this construction of $\mathbb{R}$ as inferior in some sense and they adopt the Dedekind cut. The decimal construction of $\mathbb{R}$ rather is an existence proof and the Dedekind cut an abstraction at a later phase of theory. For the

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After which he also thinks that he has proven that $\mathcal{R}$ is countable. I have exchanged dozens of e-mails about this point with Thomas Cool, and all that comes out of this exchange is that:

- he says he understands the classical definition but does not like it
- he says that he does have the right to introduce a different definition
- he says that all he wants to do is to introduce differential calculus in the secundary school without the classical definitions of limits

I showed him the "topological proof" that $\mathcal{R}$ is uncountable. He cannot find an error in it but he says in the end that he does not like it, because he feels it cannot be right.

4 Motivation

If someone writes about mathematics in the popular press or on the internet, he (or she) certainly will be seen as a kind of authority by the many people that are not very skilled in mathematics. Of course we can ignore a person that writes pseudo mathematics and presents himself as an expert in the field. But that does not help the general public to get a better understandig of what mathematics is.

Mathematicians should not write a favourable review about a paper or a book because we "see a few interesting ideas" when there are also so many mistakes and misunderstandings in the same paper. We should first of all point out what the mistakes are (especially when there are so many).

About the author

Jan van Rongen (b. 1948) has an honours degree in Mathematics from Leiden University in 1972. He worked as a post graduate until the summer of 1976 after which he pursued a career in ICT. He is now retired.

References

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